Modeling of softsphere normal collisions with characteristic of coefficient of restitution dependent on impact velocity

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Abstract This letter presents a theoretical model of the normal (head-on) collisions between two soft spheres for predicting the experimental characteristic of the coefficient of restitution dependent on impact velocity. After the contact force law between the contacted spheres during a collision is phenomenologically formulated in terms of the compression or overlap displacement under consideration of an elastic–plastic loading and a plastic unloading subprocesses, the coefficient of restitution is gained by the dynamic equation of the contact process once an initial impact velocity is input. It is found that the theoretical predictions of the coefficient of restitution varying with the impact velocity are well in agreement with the existing experimental characteristics which are fitted by the explicit formula. [doi:10.1063/2.1302103]

Keywords soft spheres, normal collisions, coefficient of restitution, impact velocity, theoretical model

The phenomena of collision between two or more bodies are one of essential dynamics and they extensively exist in nature and engineering, e.g., crater impacted by an aerolite, pounding of structures at a joint induced by a strong earthquake, and armour-piercing behavior of armament design, etc.[1–5] The investigation for a rational description of collision or impact phenomena can be traced back to three hundred years ago when the science of mechanics birthed.[6] At the early investigation, the collision was mainly treated as the hard spheres. In other words, the deformation resulted in the two collision spheres is not considered. In such case, a parameter named by the coefficient of restitution should be introduced in the investigation to characterize the energy dissipation mainly induced by the plastic deformation during the collision, which is defined by the ratio of the end velocity (or the relative velocity at the end instant of the collision) to the impact velocity (or the relative velocity at initial instant of the collision) for the normal collisions. At present, this investigation is attributed to the category of hardcollision model, while the coefficient of restitution in the applications is usually pre-specified by a constant in the region $[0, 1]$ on the basis of the experimental results of the collision systems.

Since 40–50's of the 20th century, however, the collision experiments have exhibited the characteristic that the coefficient of restitution varies with the impact velocity even if a collision system is specified, i.e., the measurements indicate that the coefficient of restitution decreases with the impact velocity.[6–10] In order to account for this characteristic in a normal collision, the deformation should be taken into account in the collision study, which is recently attributed to the category of the softcollision model. In such study, the main goal is to find a contact force law for describing the dynamic process during a normal collision such that the dependence of the coefficient of restitution on the impact velocity can be predicted out by means of the dynamic equations on the basis of the model, where the contact force is instantly dependent on its deformation state of the contacted spheres, which is similar to the results given in the Hertz contact mechanics for the case of perfectly elastic contact of two spheres.[11] As the discrete-element method (DEM) is recently employed in the numerical analysis of widespread problems of macroscopic particle accumulation or flows, e.g., windblown sand movements, sediment transportation by water, debris flow, and soil or rock mechanics, etc.,[12–16] it is found that the model for describing the contact force pays a central role in this code. At present, the discrete bodies in the DEM are mostly simplified by spheres if it is possible, and the force law is usually, for the reason of simplicity, modeled by a linear spring and a linear dashpot (i.e., it is called as the linear force model), where the energy dissipation generated by the inelastic deformation of the contacted bodies is behaved by the equivalent dashpot. Following the dynamic theory of the collisions under this model, one can get the damping coefficient ξ, and the collision duration T, respectively expressed in the form

$$\xi = -\frac{\ln\varepsilon}{\sqrt{\pi^2 + (\ln\varepsilon)^2}}, \quad T = \frac{\pi}{\sqrt{1-\xi^2}\omega_0},$$

here, $\omega_0 = \sqrt{\alpha/M}$, ε is the coefficient of resititution, α and β are the rigidity and damping constants in the linear force model, M is the effect mass. For the collisions of almost perfect plasticity or $\varepsilon \to 0$, these results tell us that we have $\xi \to 1$, further $T \to \infty$. It is obvious that the result of $T \to \infty$ is unacceptable to the case of perfectly plastic collisions.[17] When the two contacted spheres during a collision are of a perfect elasticity, i.e., the coefficient of restitution is equal to 1, on the other hand, the Hertz solution tells us

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that the spring is inherently nonlinear, i.e., the contact force is proportional to 3/2 power of the overlap displacement.[11] Hence, the linear force model has some innate defects in describing the force law of two collision or contact spheres. In order to reveal this force law, some experimental and theoretical investigations have been conducted.[10,18–24] According to treatment of the energy dissipation in the normal collision, the existing models can be divided into two categories: viscoelastic and elastic–plastic models.[8,10,18,21–23] In the viscoelastic models, the energy dissipation in the inelastic collisions is equivalent to one induced by a dashpot while the spring is sometimes revised by a nonlinear one like the Hertz solution.[2] In this kind of models, the coefficient of restitution should be pre-specified to reflect the energy loss in a normal collision. The elastic–plastic models, however, account for the energy dissipation directly by means of the plastic deformation, as a result, the coefficient of restitution is an output of the models. In order to evaluate the efficiency of a theoretical model of the force law, the prediction of the changeable coefficient of restitution becomes one of the essentials. On the basis of the experimental data, Stevens and Hrenya[10] studied the suitability of seven existing models by comparing their predictions with the experimental results, from which none of the models is found to give a satisfactory prediction of the characteristic dependence of the coefficient of restitution. Meanwhile, it is also known that the quantitative predictions from the existing models differ significantly.[10] When the impact–velocity–dependent coefficient of restitution was taken into account in the simulation of a vibrated granular medium, for example, the experimental results of coefficient of restitution were chosen from the case of steel plate impacted by a sphere,[25] the simulation results exhibit some significant difference when a constant coefficient of restitution is taken by a small discrepancy.

Here, we report a phenomenological model of the force law for the normal collisions under consideration of the elastic–plastic loading process and the plastic unloading process during a normal collision on the basis of the issue of the Hertz contact solution, in which all parameters appeared in the model are either of geometry or of material characteristics determined by those feasible experiments.

According to the Hertz contact solution for the elastic case,[11] we have the following relations

$$P = k\delta^{3/2}, \quad a = bk^{1/3}\delta^{1/2}, \quad q_{\max} = \frac{3k\delta^{3/2}}{2\pi a^2}, \tag{1}$$

here, P and δ stand for the contact force and the relative approaching (or overlap) displacement between two contacted spheres, a is the contact radius, $q_{\max}$ represents the maximum pressure at the center of contact area, and the parameters b and k are formulated by the material and geometric constants of the form

$$b = \left(\frac{3\pi R}{4E}\right)^{1/3}, \quad k = \frac{4}{3}ER^{1/2}, \tag{2}$$

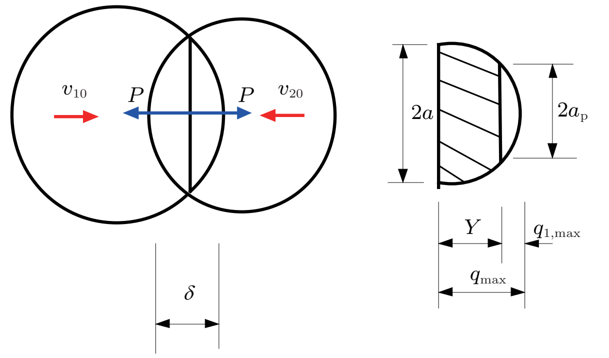

Fig. 1. Schematic drawing of compression state of two collision spheres.

in which $E = [(1-\mu_1^2)/E_1 + (1-\mu_2^2)/E_2]^{-1}$ and $R = R_1R_2/(R_1+R_2)$ indicate the effective Young's modulus and the effective radius of the collision spheres, respectively. Here, E_i, μ_i and R_i $(i = 1, 2)$ are Young's modulus, Poison's ratio and radius of i-th collision body, respectively.

For the collision problem, the dynamic equation and initial conditions may be respectively formulated as[10,17,18]

$$M\ddot{\delta} = -P, \tag{3}$$

$$t = 0: \quad \delta = 0, \quad \dot{\delta} = \dot{\delta}_0 > 0, \tag{4}$$

where $M = M_1M_2/(M_1+M_2)$ indicates the effective mass, $M_i (i = 1, 2)$ is the mass of i-th body, and $\dot{\delta}_0$ represents the impact velocity. Denote the minimum yield strength of the collision spheres by Y. Then we know that the collision sphere with the yield strength Y will enter plastic deformation when $q_{\max} \geqslant Y$. From the Hertz elastic contact, we obtain the critical elastic displacement $\delta_{\rm cr}^{\rm e}$ and the critical impact velocity $\dot{\delta}_{\rm cr}^{\rm e}(\dot{\delta}_{\rm cr}^{\rm e} = \dot{\delta}_{0,\rm cr}^{\rm e} = \dot{\delta}_0(q_{\max})|_{q_{\max}=Y})$ in the form

$$\delta_{\rm cr}^{\rm e} = \left(\frac{5M}{4k}\right)^{2/5}(\dot{\delta}_{0,\rm cr}^{\rm e})^{4/5} = \frac{\pi^{10/3}}{4}\left(\frac{Y}{E}\right)^2 R, \tag{5a}$$

$$\dot{\delta}_{0,\rm cr}^{\rm e} = \left(\frac{\pi^{22/3}E}{160\rho}\right)^{1/2}\left(\frac{Y}{E}\right)^{5/2}, \tag{5b}$$

here, ρ is the density of mass and it satisfies the relation, $M = 4\pi\rho R^3/3$. For the Hertz contact solution, we have the condition $0 \leqslant \dot{\delta}_0 \leqslant \dot{\delta}_{\rm cr}^{\rm e}$ when the Hertz solution is of validity.

When the impact velocity is over the critical velocity $\dot{\delta}_{0,\rm cr}^{\rm e}$ of fully elastic deformation in the collision system, we know that one of the collision bodies enters in plastic deformation. In such case, we still denote the resultant contact force and displacement by P and δ, respectively. For the considered case here, we denote the radius of circular area of plastic region in the contact region by $a_{\rm p}$ (see Fig. 1).

According to the similarity of distribution force at the contact surface, we have $a_{\rm p} = \sqrt{\delta^2 - (\delta_{\rm cr}^{\rm e})^2}$. Taking the integral calculation to the force distribution in the contact area, we get the loading force expressed by the displacement of the form

$$P_{\mathrm{L}} = k\delta^{3/2} - \left(\int_0^{a_{\mathrm{p}}} 2\pi x q_{\max}(\delta) \sqrt{1 - \left(\frac{x}{a}\right)^2}\, \mathrm{d}\, x - \pi a_{\mathrm{p}}^2 Y \right) = k\delta^{3/2} \cos^3\theta + \pi[\delta^2 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2]Y, \tag{6}$$

in which

$$\theta = \sin\frac{a_{\mathrm{p}}}{a} = \sin\left(\frac{\sqrt{\delta^2 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2}}{bk^{1/3}\delta^{1/2}}\right), \tag{7}$$

when $\delta = \delta_{\max}$ at the end of loading subprocess, where $\delta_{\max}$ is the maximum value of the overlap displacement, we denote $a_{\mathrm{p,max}} = \sqrt{\delta_{\max}^2 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2}$. Considering $bk^{1/3} \sim R^{1/2} \gg \delta^{1/2}$, we can take an expansion to the term of $\cos^3\theta$, then the load-displacement relation expressed in Eq. (6) is reduced into the form

$$P_{\mathrm{L}} \approx k\delta^{3/2} - \frac{3k^{1/3}}{2b^2}\delta^{5/2} + \frac{3k^{1/3}(\delta_{\mathrm{cr}}^{\mathrm{e}})^2}{2b^2}\delta^{1/2} + \pi Y[\delta^2 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2]. \tag{8}$$

Substituting Eq. (8) into Eq. (3) then taking the energy integration to the resulting dynamic equation from the initial instant with the state $\delta = 0$ and $\dot{\delta} = \dot{\delta}_0$ to the end instant with the state $\delta = \delta_{\max}$ and $\dot{\delta} = 0$, we have the following energy relation

$$\frac{2k}{5M}\delta_{\max}^{5/2} - \frac{3k^{1/3}}{7b^2}\left[\delta_{\max}^{7/2} - (\delta_{\mathrm{cr}}^{\mathrm{e}})^{7/2}\right] + \frac{k^{1/3}}{b^2}\left[\delta_{\max}^{3/2} - (\delta_{\mathrm{cr}}^{\mathrm{e}})^{3/2}\right] + \pi Y\left\{\frac{1}{3}[\delta_{\max}^3 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^3] - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2(\delta_{\max} - \delta_{\mathrm{cr}}^{\mathrm{e}})\right\} = \frac{1}{2}M\dot{\delta}_0^2. \tag{9}$$

Let $\delta_{\max} = x_{\max}\delta_{\mathrm{cr}}^{\mathrm{e}}$, $\dot{\delta}_0 = y\dot{\delta}_{0,\mathrm{cr}}^{\mathrm{e}}$ $(y \geqslant 1)$. Substitution of them into Eq. (9) yields the following nonlinear algebraic equation in the dimensionless form

$$\frac{2}{5}x_{\max}^{5/2} - c_1(x_{\max}^{7/2} - 1) + c_2(x_{\max}^3 - 1) - c_3(x_{\max}^{-}1) + \frac{1}{7}c_1(x_{\max}^{3/2} - 1) = \frac{2}{5}y^2, \tag{10}$$

here, $c_1 = 2\kappa/21$, $c_2 = 2\kappa/9$, $c_3 = 2\kappa/3$, and $\kappa = \pi^2 b^2 Y^2/k^{4/3} = 36(\pi/2)^{8/3}(Y^2/E^2) \sim (Y/E)^2$.

In the collision between two spheres, there is a restitution subprocess within that the contact force is unloaded. In this subprocess, we take a straight line depending on the overlap displacement to behave for the plastic unloading force except for the elastic part. Considering the continuous condition of loading and unloading forces at the state $\delta = \delta_{\max}$, i.e., $P_{\mathrm{L}}|_{\delta=\delta_{\max}} = P_{\mathrm{U}}|_{\delta=\delta_{\max}}$, we have the unloading force law

$$P_{\mathrm{U}} = k\delta^{3/2} - \int_0^{a_{\mathrm{p}}} 2\pi r q_{\max}(\delta)\sqrt{1 - (r/a)^2}\, \mathrm{d}\, r + \pi a_{\mathrm{p,max}}^2 Y\left[1 - \tilde{\chi}\frac{E}{Y}\left(1 - \frac{\delta}{\delta_{\max}}\right)\right] = k\delta_{\max}^{3/2}\cos^3\theta + \pi a_{\mathrm{p,max}}^2 Y \cdot \left[1 - \tilde{\chi}\frac{E}{Y}\left(1 - \frac{\delta}{\delta_{\max}}\right)\right], \tag{11}$$

where the term relevant to the coefficient $\tilde{\chi}$ is the plastic unloading force, and $\tilde{\chi}$ stands for a dimensionless factor of the plastic unloading. Similar to the calculations in the loading process, Eq. (11) is further reduced into

$$P_{\mathrm{U}} = k\left(1 - \frac{3\delta_{\max}}{2b^2k^{2/3}}\right)\delta^{3/2} + \frac{3(\delta_{\mathrm{cr}}^{\mathrm{e}})^2}{2b^2k^{2/3}}\delta^{1/2} + \pi Y[\delta_{\max}^2 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2]\left[1 - \tilde{\chi}\frac{E}{Y}\left(1 - \frac{\delta}{\delta_{\max}}\right)\right]. \tag{12}$$

Denote the residual displacement by δ_* at the instant when the collision is ended when $\delta = \delta_*$ and $\dot{\delta} < 0$. Then, we know that $P_{\mathrm{U}}|_{\delta=\delta_*} = 0$. Thus, we have

$$k\left(1 - \frac{3\delta_{\max}}{2b^2k^{2/3}}\right)\delta_*^{3/2} + \frac{3k^{1/3}(\delta_{\mathrm{cr}}^{\mathrm{e}})^2}{2b^2}\delta_*^{1/2} + \pi Y[\delta_{\max}^2 - (\delta_{\mathrm{cr}}^{\mathrm{e}})^2] \cdot \left[1 - \tilde{\chi}\frac{E}{Y}\left(1 - \frac{\delta_*}{\delta_{\max}}\right)\right] = 0. \tag{13}$$

Let $\delta_* = x_*\delta_{\mathrm{cr}}^{\mathrm{e}}$. Then Eq. (13) is simplified by the following dimensionless form

$$(1 - d_1 x_{\max})x_*^{3/2} + d_1 x_*^{1/2} + d_2(x_{\max}^2 - 1) \cdot \left[1 - \tilde{\chi}\frac{E}{Y}\left(1 - \frac{x_*}{x_{\max}}\right)\right] = 0, \tag{14}$$

here $d_1 = d_2 = 2\kappa/3$.

Taking the calculation for the energy principle to the unloading process from $\delta = \delta_{\max}$ to $\delta = \delta_*$, and considering the definition of coefficient of restitution, i.e., $\varepsilon = -\dot{\delta}_{\mathrm{f}}/\dot{\delta}_0$, here $\dot{\delta}_{\mathrm{f}}$ indicates the relative velocity between two spheres at the end of collision, i.e., $\dot{\delta}_{\mathrm{f}} = \dot{\delta}|_{t=T}$, we can get the formula of coefficient of restitution relevant to the plastic unloading in the form

$$\varepsilon_{\mathrm{p}} = \frac{1}{y}\sqrt{\frac{5}{2}}\left\{\frac{2}{5}(1 - h_1 x_{\max})(x_{\max}^{5/2} - x_*^{5/2}) + h_2(x_{\max}^2 - 1)\left[\left(1 - \tilde{\chi}\frac{E}{Y}\right)(x_{\max} - x_*) - \frac{\tilde{\chi}E}{2Y}\left(x_{\max} - \frac{x_*^2}{x_{\max}}\right)\right] + h_3(x_{\max}^{3/2} - x_*^{3/2})\right\}^{1/2}, \tag{15}$$

in which $h_1 = h_2/2 = h_3/2 = 2\kappa/3$. To many materials, we have $Y/E \sim 10^{-1} \sim 10^{-2}$, further, $h_1 \sim h_2 \sim \kappa \sim 10^{-2} \sim 10^{-4} \ll 1$. For the case of perfect elasticity when $x_{\max} = 1$, $x_* = 0$ and $y = 1$, we get the result of $\varepsilon_{\mathrm{p}} \approx 1$ from Eq. (15). Here, the subscript "p" represents the quantity relevant to plastic deformation.

Table 1. Essential parameters and constants of the collision experiments.

Parameters	Stainless steel	Chrome steel	Nylon
Young's modulus $E_1 = E_2/(\mathrm{N\cdot m^{-2}})$	1.93×10^{11}	2.03×10^{11}	2.5×10^{9} [a]
Poison's ratio $\mu_1 = \mu_2$	0.35	0.28	0.30
Mass density $\rho_1 = \rho_2/(\mathrm{kg\cdot m^{-3}})$	8 030	7 830	1 140
Yield strength $Y/(\mathrm{N\cdot m^{-2}})$	3.10×10^{8}	2.03×10^{9}	4.0×10^{7}
Radius $R_1 = R_2/\mathrm{m}$	0.012 7	0.012 7	0.006 35, 0.012 7, 0.025 4
Critical impact velocity $\dot{\delta}^{\mathrm{e}}_{\mathrm{cr}}/(\mathrm{m\cdot s^{-1}})$	0.008 2	0.909 0	0.835 2
Limit impact velocity $\dot{\delta}_{\mathrm{lim}}/(\mathrm{m\cdot s^{-1}})$	340	882	324
Coefficient σ	3.043 2	0.141 8	0.435 5
Coefficient ψ	0.540 0	0.311 4	0.541 4
Ratio of E/Y	354.75	54.25	34.34
Unloading factor $\tilde{\chi}$	0.136 8	0.023 9	0.016 0

[a] Here, the Young's modulus of the nylon spheres is taken from the value of bulk nylon from the web site: http://www-materials.eng.cam.ac.uk/mpsite/short/OCR/ropes/default.html, where the value of this parameter is about one order higher than one given in Ref. 20.

From the experimental measurement of coefficient of restitution,[10] we know that the coefficient of restitution is less than 1 when $\dot{\delta}_0 < \dot{\delta}^{\mathrm{e}}_{\mathrm{cr}}$, which implies that there is some energy dissipation in the collision system even when the collision system has only elastic deformation. In fact, this energy dissipation is relevant to some sound and heat energies. To behave for the energy dissipation irrelevant to the plastic deformation, we denote the part of coefficient of restitution by $\varepsilon_{\mathrm{up}}$ which corresponds to those dissipation part irrelevant to the plastic deformation. From the knowledge of collision physics, we can denote $\varepsilon_{\mathrm{up}} = \varepsilon|_{y=1}$ without losing generality, and the energy dissipation irrelevant to the plastic deformation may be formulated by $\Delta T = (1-\varepsilon^2_{\mathrm{up}})M\dot{\delta}^2_0/2$. Thus the coefficient of restitution ε can be expressed by

$$\varepsilon = \left\{\frac{5}{2y}\left\{\frac{2}{5}(1-h_1x_{\max})(x^{5/2}_{\max}-x^{2/5}_{*}) + h_2(x^2_{\max}-1)\left[\left(1-\tilde{\chi}\frac{E}{Y}\right)(x_{\max}-x_*) - \frac{\tilde{\chi}E}{2Y}\left(x_{\max}-\frac{x^2_*}{x_{\max}}\right)\right] + h_3(x^{3/2}_{\max}-x^{3/2}_*)\right\} - 1+\varepsilon^2_{\mathrm{up}}\right\}^{1/2} \tag{16}$$

when $y \geqslant 1$. It is obvious that when $x_{\max} = 1$, we have $\varepsilon \approx \varepsilon_{\mathrm{up}}$. It is obvious that $\varepsilon_{\mathrm{up}}$ must be determined by the experimental measurement to which the empirical formula is employed[17]

$$\varepsilon = \exp\{-\sigma[\dot{\delta}_0/(\dot{\delta}_{\mathrm{lim}}-\dot{\delta}_0)]^{\psi}\}, \quad \dot{\delta}_0 \geqslant \dot{\delta}^{\mathrm{e}}_{\mathrm{cr}} \tag{17}$$

here, σ and ψ are the fitting coefficients, and $\dot{\delta}_{\mathrm{lim}}$ is the limit impact velocity corresponding to the case when $\varepsilon = 0$. An estimation of $\dot{\delta}_{\mathrm{lim}}$ is given by the formula of $\dot{\delta}_{\mathrm{lim}} \approx \sqrt{3Y/\rho}$ where ρ is the mass density of the sphere to which the sphere enters the plastic state with the yield strength Y. After that, we have $\varepsilon_{\mathrm{up}} = \exp\{-\sigma[\dot{\delta}_0/(\dot{\delta}_{\mathrm{lim}}-\dot{\delta}_0)]^{\psi}\}|_{\dot{\delta}_0=\dot{\delta}^{\mathrm{e}}_{\mathrm{cr}}}$.

The remain one parameter in the theoretical model is to select the unloading factor, which can be determined by some essential experiments like other elastic constants. Here, we do it on the basis of the experimental measurements of the materials of stainless steel and chrome steel spheres conducted by Stevens and Hrenya.[10]

Due to the coefficient of restitution dependent on the ratio of E/Y, it is suitably assumed that unloading factor varies with the ratio too. For the simplest case, we take the linear relation $\tilde{\chi} \sim E/Y$, i.e.

$$\tilde{\chi} = c_1 + c_2\frac{E}{Y}, \tag{18}$$

where c_1 and c_2 are constants to be determined. Applying the data of two kinds of experiments of stainless steel and chrome steel spheres dealt with by Eq. (17), we get $c_1 = 3.518\times10^3$ and $c_2 = 3.757\times10^{-4}$.

The parameters appeared in the theoretical model or formulae are listed in Table 1. Figure 2 displays the comparison of theoretical predictions of the coefficient of restitution for the three collision materials and their experimental data formulated by the empirical formula of Eq. (17). Here, the curves marked by "theory of Thornton" are the predictions of the model in Ref. 18. It is found from them that the predictions of this collision model quantitatively agree with the experimental results of Eq. (17) in the whole region of impact velocity. For the case of nylon spheres with smallest radius of 0.006 35 m when the dimensionless velocity y is greater than 10, it is found that the practical measurements are lower than both the theoretical predictions and empirical output notably (see Fig. 2). This difference is possibly generated by the data processing in its experiment. With the limit of space, here, we neglect the detail reason why it is.

Thus, the theoretical model proposed in this letter is successfully established to predict the coefficient of restitution in a normal collision between two spheres.

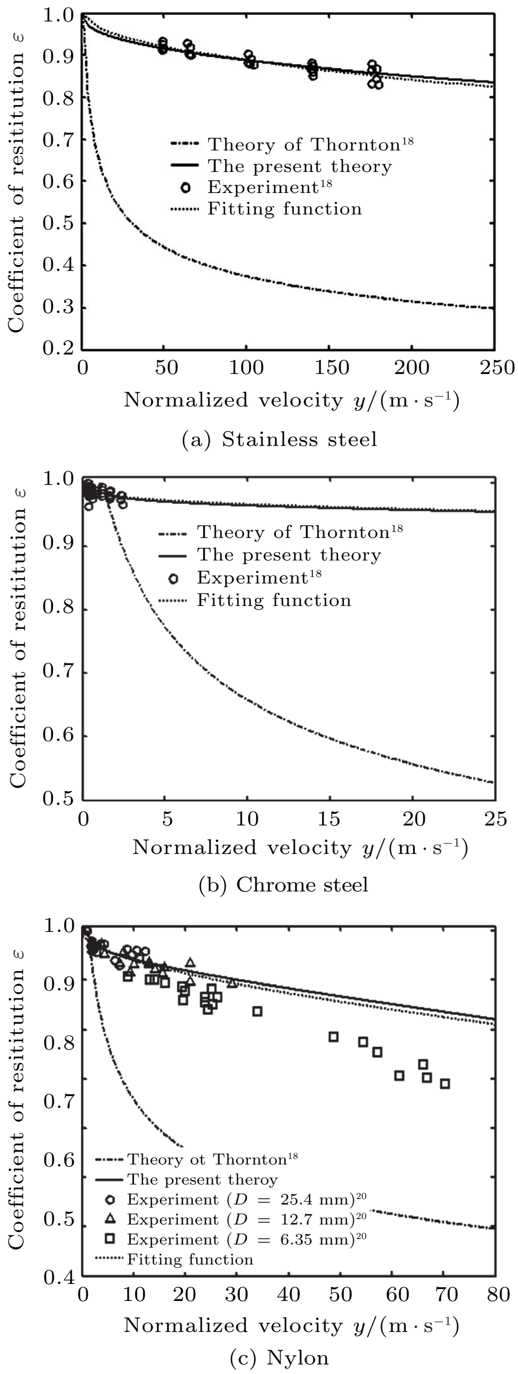

Fig. 2. Theoretical predictions of coefficient of restitution varying with the impact velocity ($y = \dot{\delta}_0/\dot{\delta}^{\mathrm{e}}_{0,\mathrm{cr}}$) are compared with the experimental data for the collision of two spheres.

The predictions of the characteristic of coefficient of restitution varying with the impact velocity in a normal collision display that they are quantitatively in agreement with the experimental measurements well.

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